Quantum Algorithms III

The Quantum Computing Stack

5	Applications
3	Algorithms
2	Circuits
4	Hardware
1	Mechanics

Grovers!
- useful
- Quadratic speedup

Unordered Search

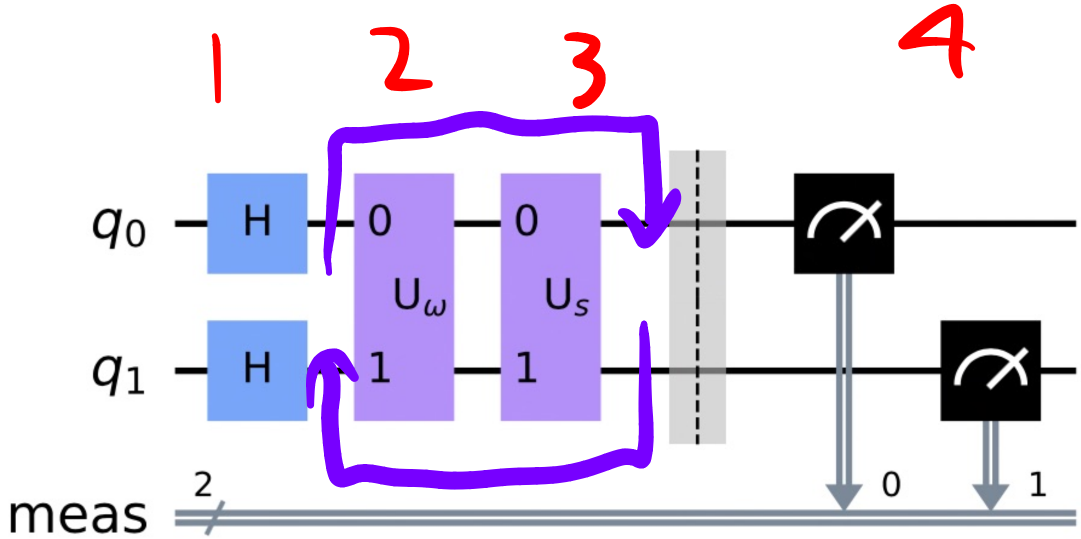
6	22	1	8	4	5	10	12

elements

Classically: $O(N)$

Quantum: $O(\sqrt{N})$

Outline of Grover's



Demo!

1. Superposition
2. Oracle selects solution
3. Amplify solution
4. Measure outcome

repeat $\sim \frac{\pi}{4}\sqrt{N}$ times

Reflections

In General

$$R_\psi = 2|\psi\rangle\langle\psi| - I$$

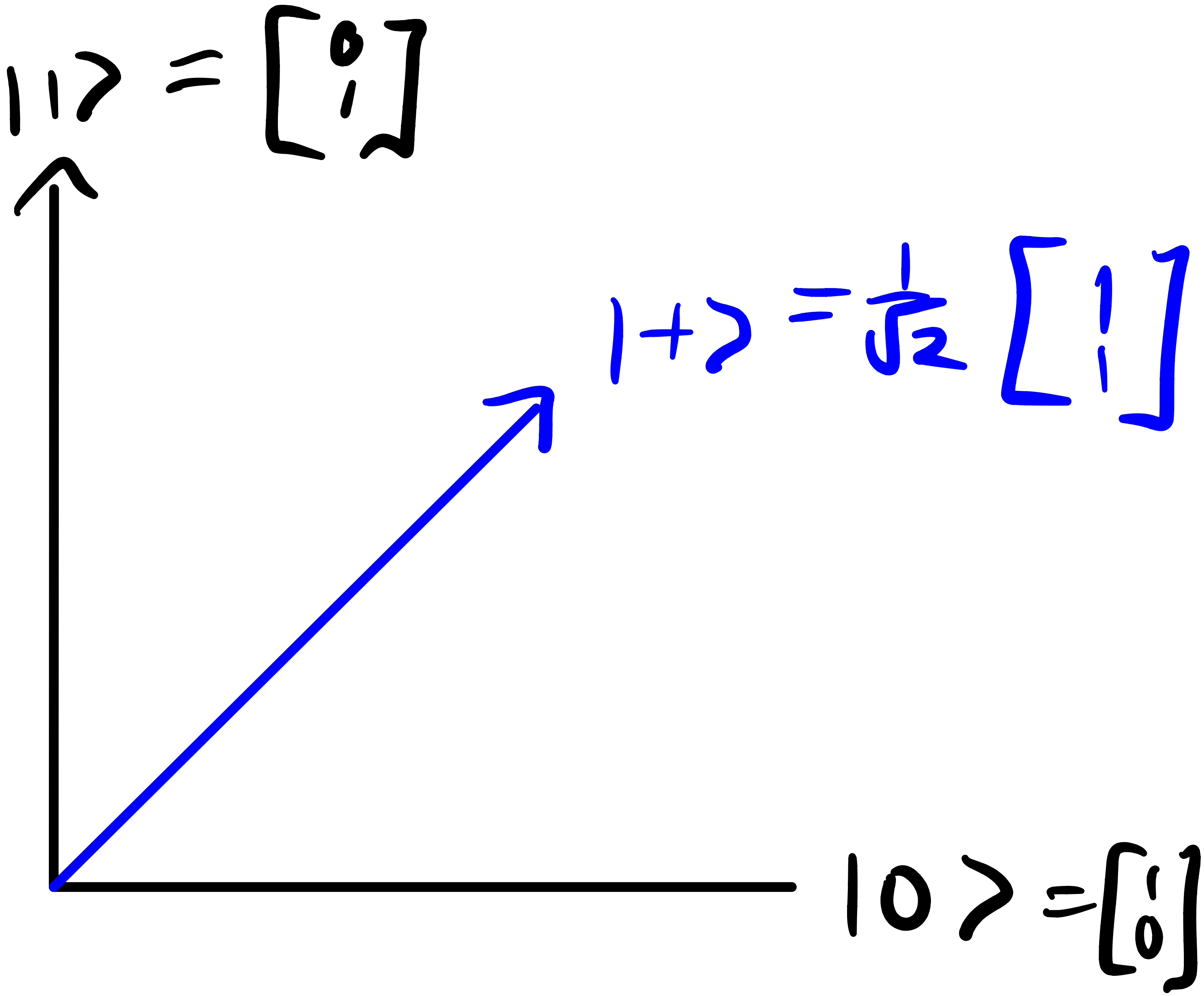
Example

$$|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$R_+ = 2\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$



$$R_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

Reflections are Unitary?

$$UU^\dagger = I$$

$$R_\psi^\dagger = (2|\psi\rangle\langle\psi| - I)^\dagger$$

$$= 2|\psi\rangle\langle\psi|^\dagger - I^\dagger$$

$$= 2|\psi\rangle\langle\psi| - I$$

$$UU^\dagger = I$$

$$R_\psi R_\psi^\dagger = (2|\psi\rangle\langle\psi| - I)^2$$

$$= (2|\psi\rangle\langle\psi|)^2 + I^2 - 4|\psi\rangle\langle\psi|$$

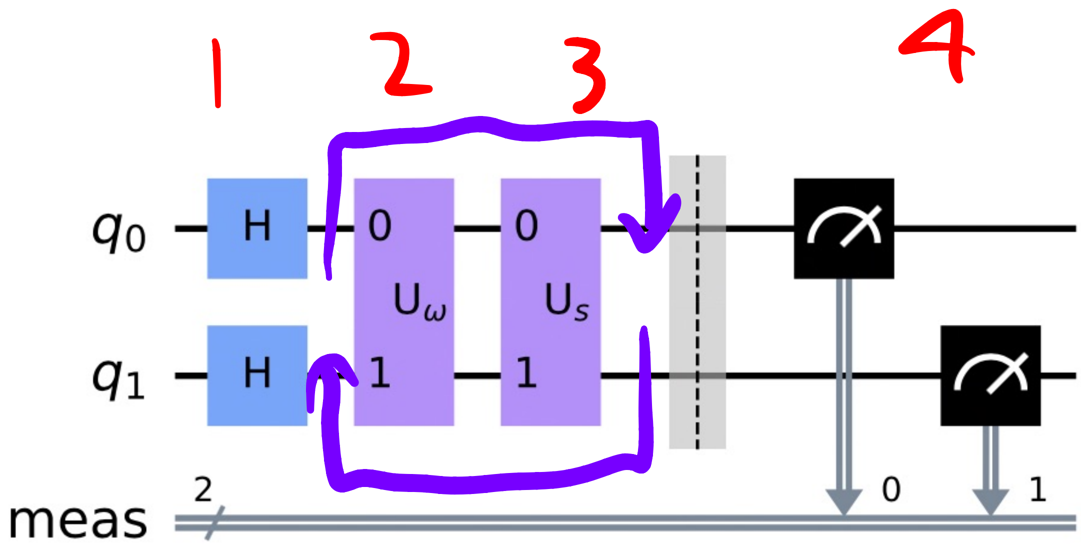
$$4|\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle\psi| = 4|\psi\rangle\langle\psi|$$

$$= 4|\psi\rangle\langle\psi| - 4|\psi\rangle\langle\psi| + I$$

$$= I$$

Unitary!

Outline of Grover's



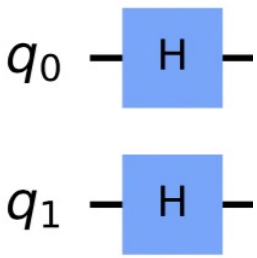
Demo!

1. Superposition
2. Oracle selects solution
3. Amplify solution
4. Measure outcome

repeat $\sim \frac{\pi}{4}\sqrt{N}$ times

1. Superposition

$N = 4$ items



$\rightarrow$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array} = |+\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$$

$n = 2$ qubits

4 items

$$|+\rangle \otimes |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} = \frac{1}{2} \begin{array}{ccc} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

Encoding

n qubits $\quad$ N items

$$2^n = N$$

"n qubits can index up to 2^n items"

$$n = \lceil \log_2 N \rceil$$

"I need $\lceil \log_2 N \rceil$ qubits to distinguish N items"

In General

$$|s\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n} |x\rangle$$

$n=2$ <u>example</u>

$$|s\rangle = \frac{1}{\sqrt{2^2}} \sum_{x=0}^{2^n} |x\rangle$$

$$= \frac{1}{2}\left(|0\rangle + |1\rangle + |2\rangle + |3\rangle\right)$$

What about for $n=3$?

Grover's Motivation

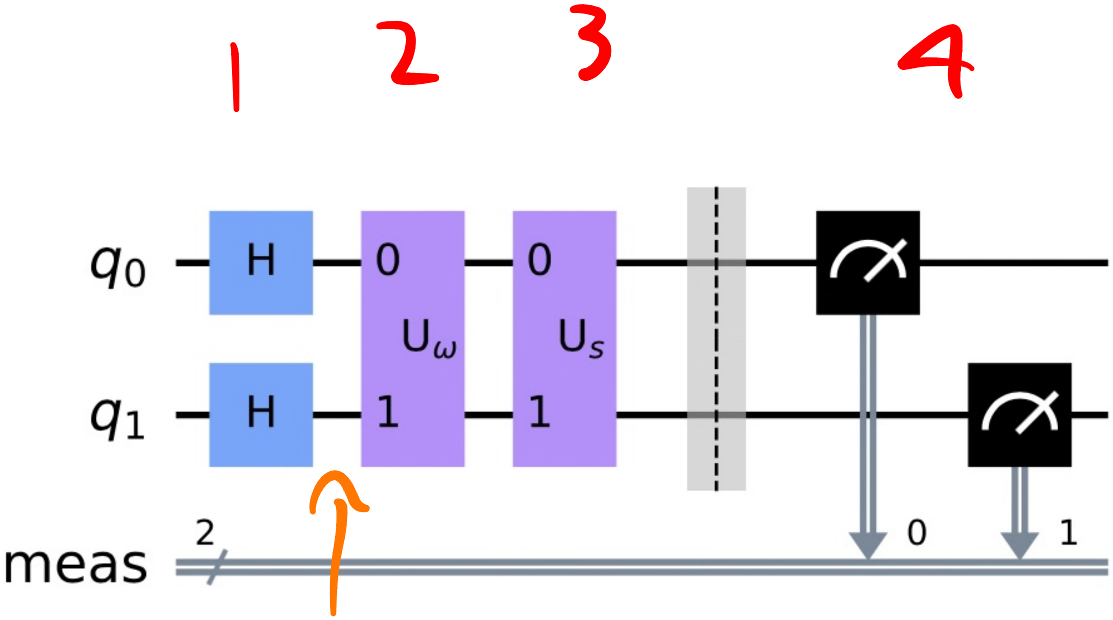
$n=2$

Starting from $|s\rangle = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

we want to measure our desired index (say $|2\rangle = |10\rangle$)

we know $|s\rangle \rightarrow |x_0\rangle$ oracle knows

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rightarrow \begin{array}{cc}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} & \begin{array}{c}0\\1\\2\\3\end{array}\end{array}$$



$$|s\rangle = \frac{1}{\sqrt{4}} \sum_{x=0}^{4} |x\rangle$$

1. Super position ✓
2. Oracle U_w selects solution
3. Amplify U_s solution
4. Measure outcome

Grover iteration

repeat $\sim \frac{\pi}{4}\sqrt{N}$ times

2. Oracle Selects Solution

- For our purposes, oracle is a heuristic.
- In practice, we don't know oracle's output. Same as Deutsch!
- For studying the Oracle, we want to pre-select an output to make sure the algorithm works properly

$$\boxed{U_w |s\rangle}$$

How the Oracle Selects a Solution

- Our search verification is $f(x) = \{ \underset{\text{True}}{1}, \underset{\text{false}}{0} \}$

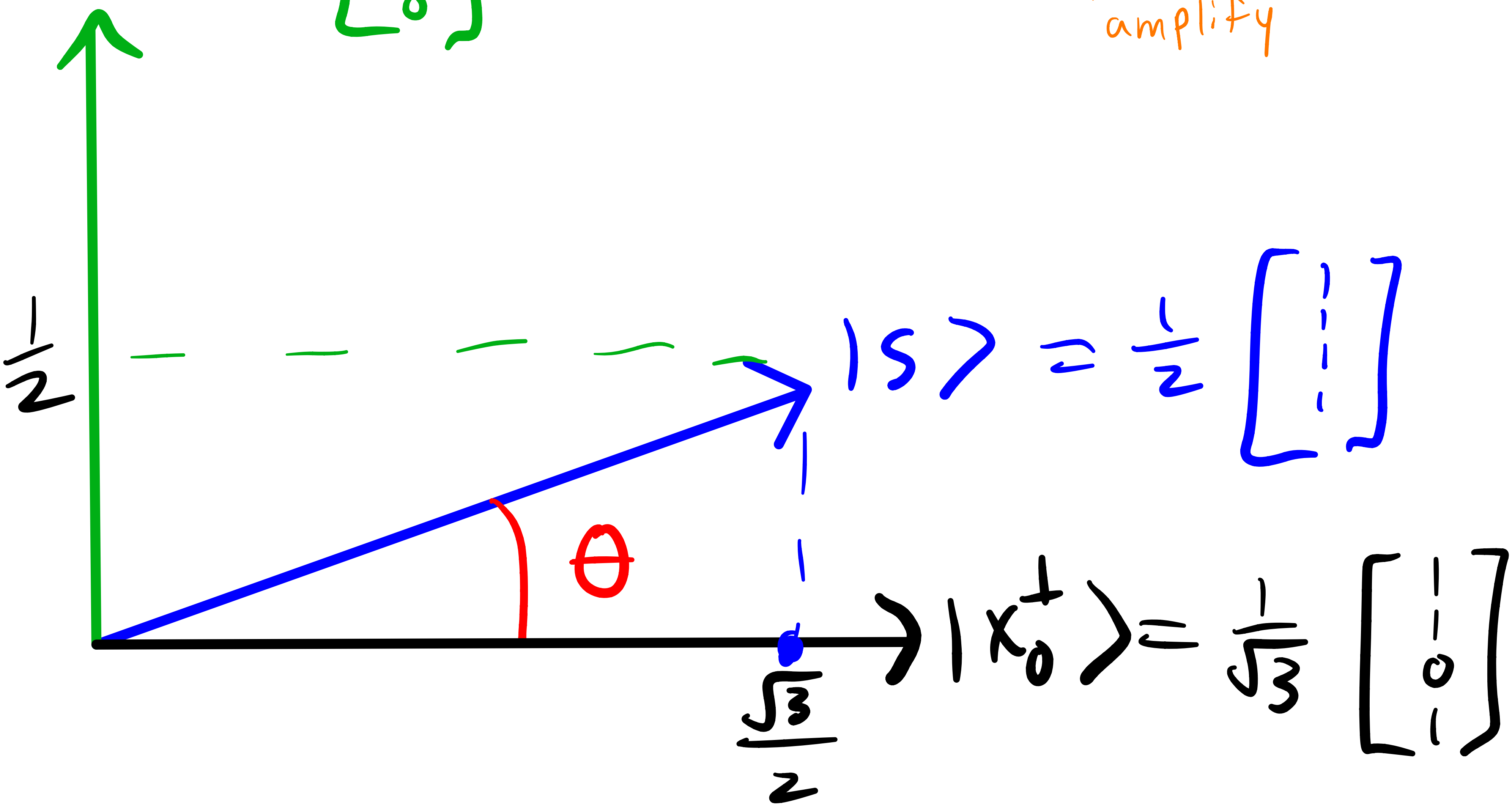
<u>Given</u> $|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

<u>selection 2</u>

$$U_w|s\rangle = \frac{1}{2}\begin{bmatrix} (-1)^{f(0)} \\ (-1)^{f(1)} \\ (-1)^{f(2)} \\ (-1)^{f(3)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

what do U_w "select" and U_s "amplify" do?

$$|x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

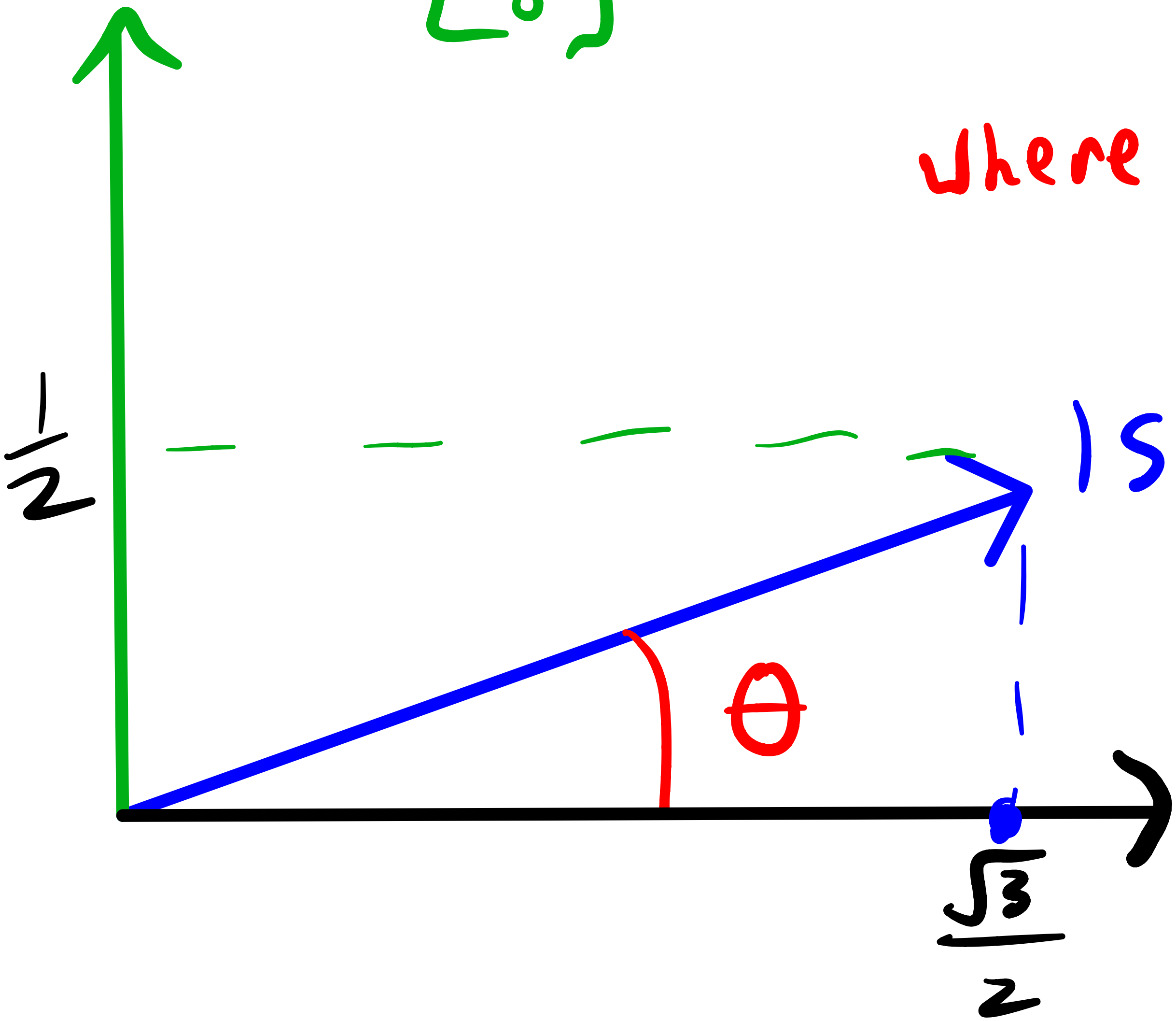


$U_w|s\rangle = ?$

$$U_w|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$|x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

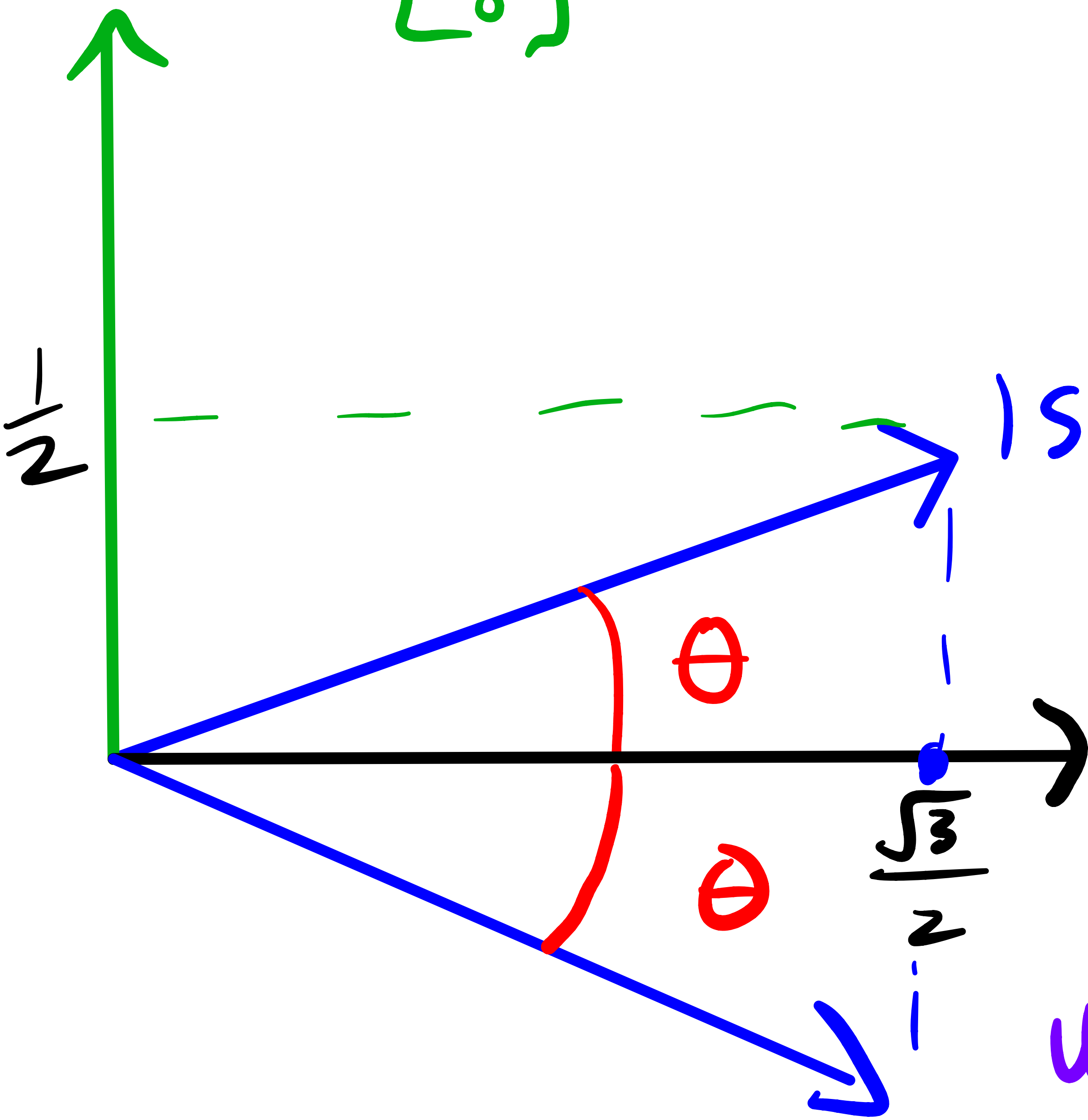
where is it?



$$|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|x_0^\perp\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$|x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

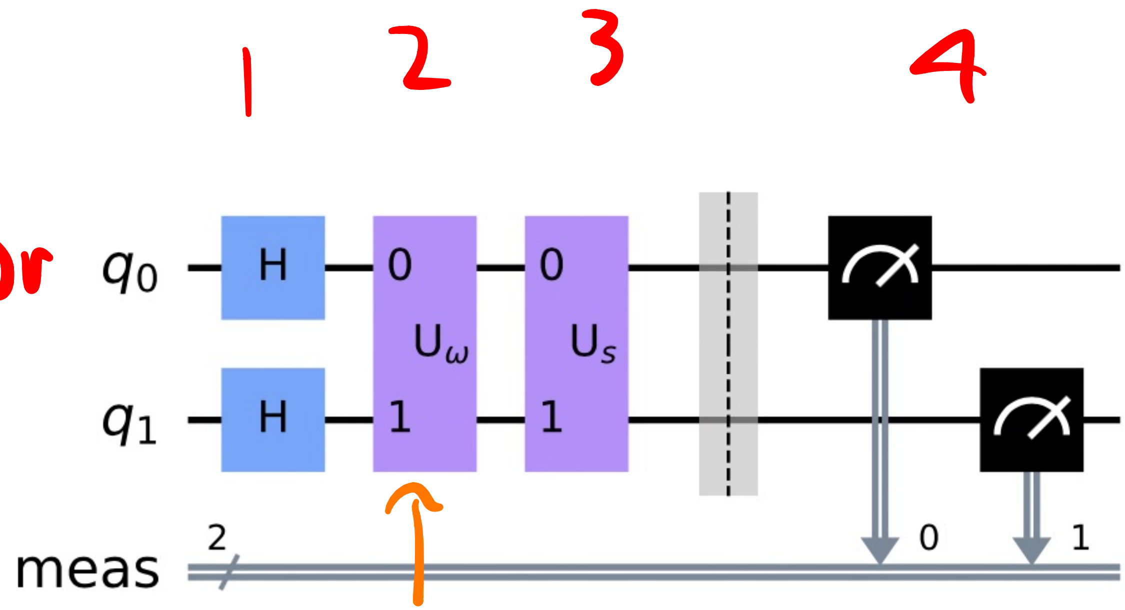


$$|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|x_0^\perp\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$U_w|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Project over orthonormal Vector by negating your original projection's reflection



~ Pulling out of thin air ~

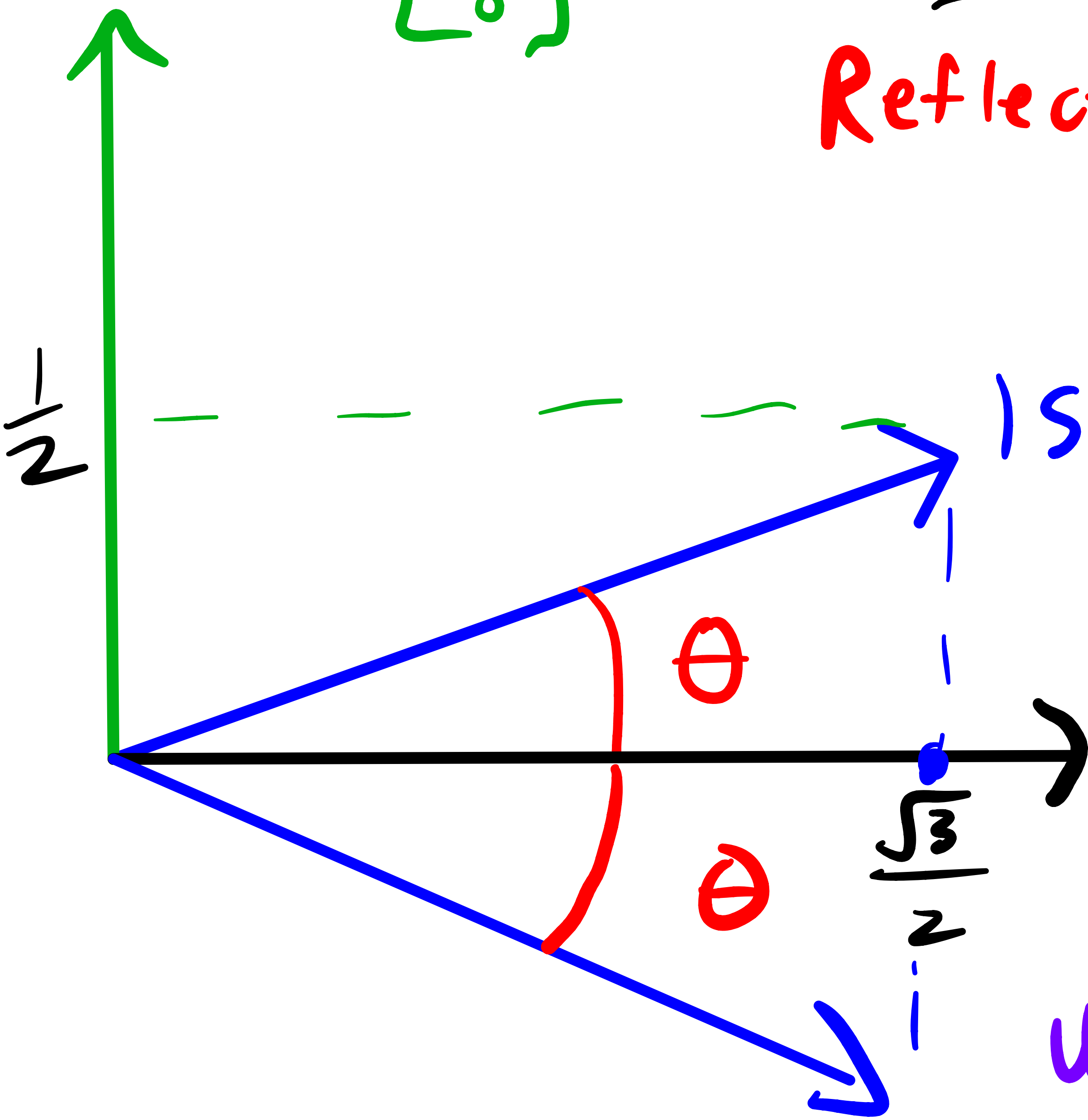
$$U_w = -\left(2|x_0\rangle\langle x_0| - I\right)$$

$$I - 2|x_0\rangle\langle x_0|$$

$$|x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$I - 2|x_0\rangle\langle x_0|$$

Reflects over $|x_0^{\perp}\rangle$



$$|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|x_0^{\perp}\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$U_w|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

exercise

$$U_w = I - 2 \underbrace{|x_0\rangle}_{\text{ket}} \underbrace{\langle x_0|}_{\text{bra}} \qquad |x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - 2 \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} & & & \end{bmatrix}$$

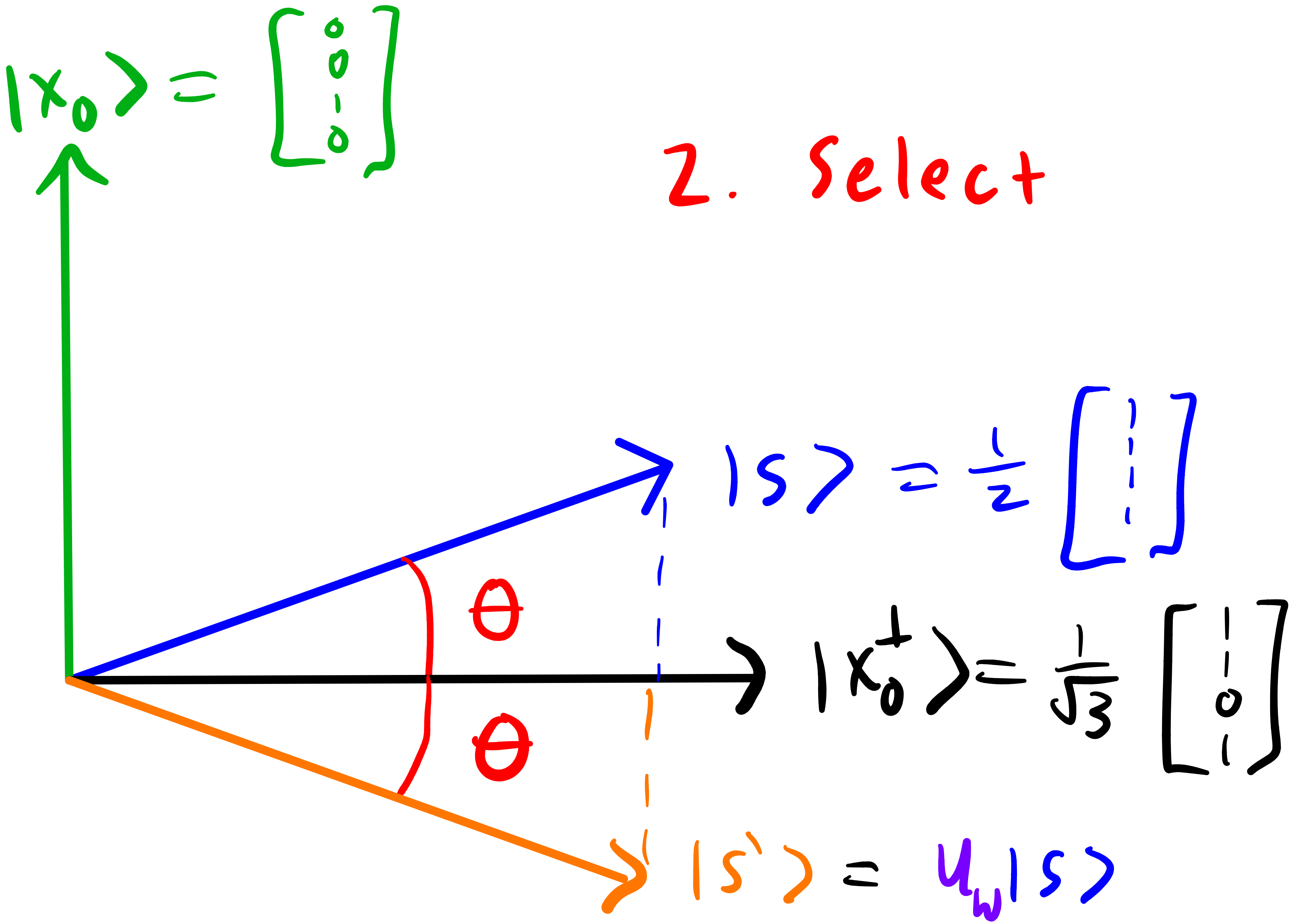
$$=$$

$$\left(I - 2|x_0\rangle\langle x_0|\right)|s\rangle$$

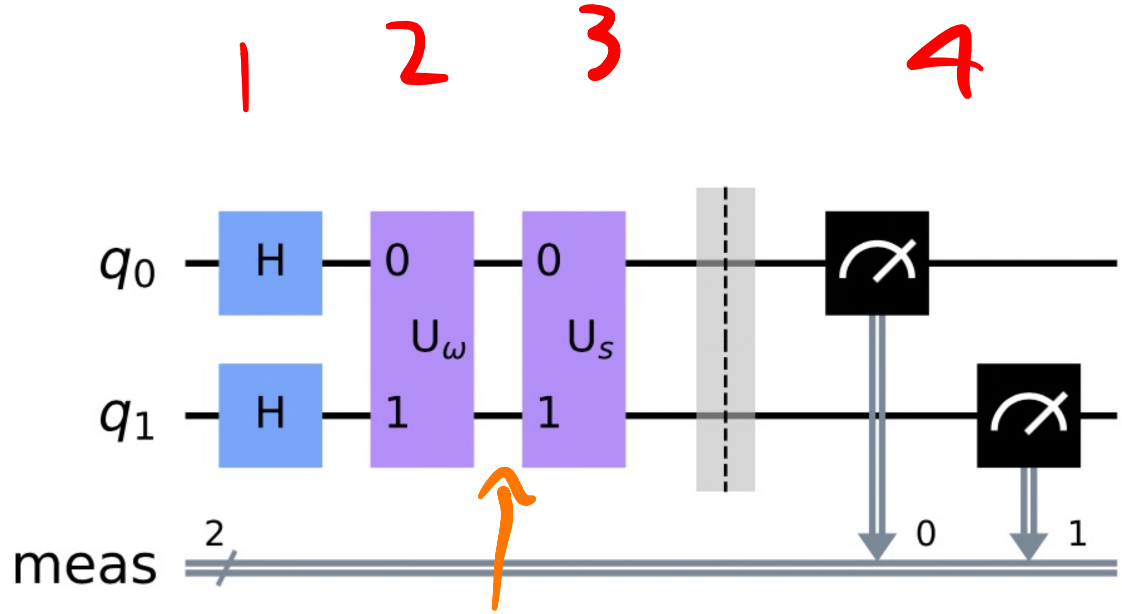
$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) |s\rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = |s'\rangle$$

2. Select



$|x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
$|s\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
θ
$|x_0^{\perp}\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$
θ
$|s'\rangle = U_w|s\rangle$



$$|s\rangle = \frac{1}{\sqrt{4}} \sum_{x=0}^{4} |x\rangle$$

1. Superposition ✓
2. Oracle U_w selects solution ✓
3. Amplify U_s solution
4. Measure outcome

Grover iteration

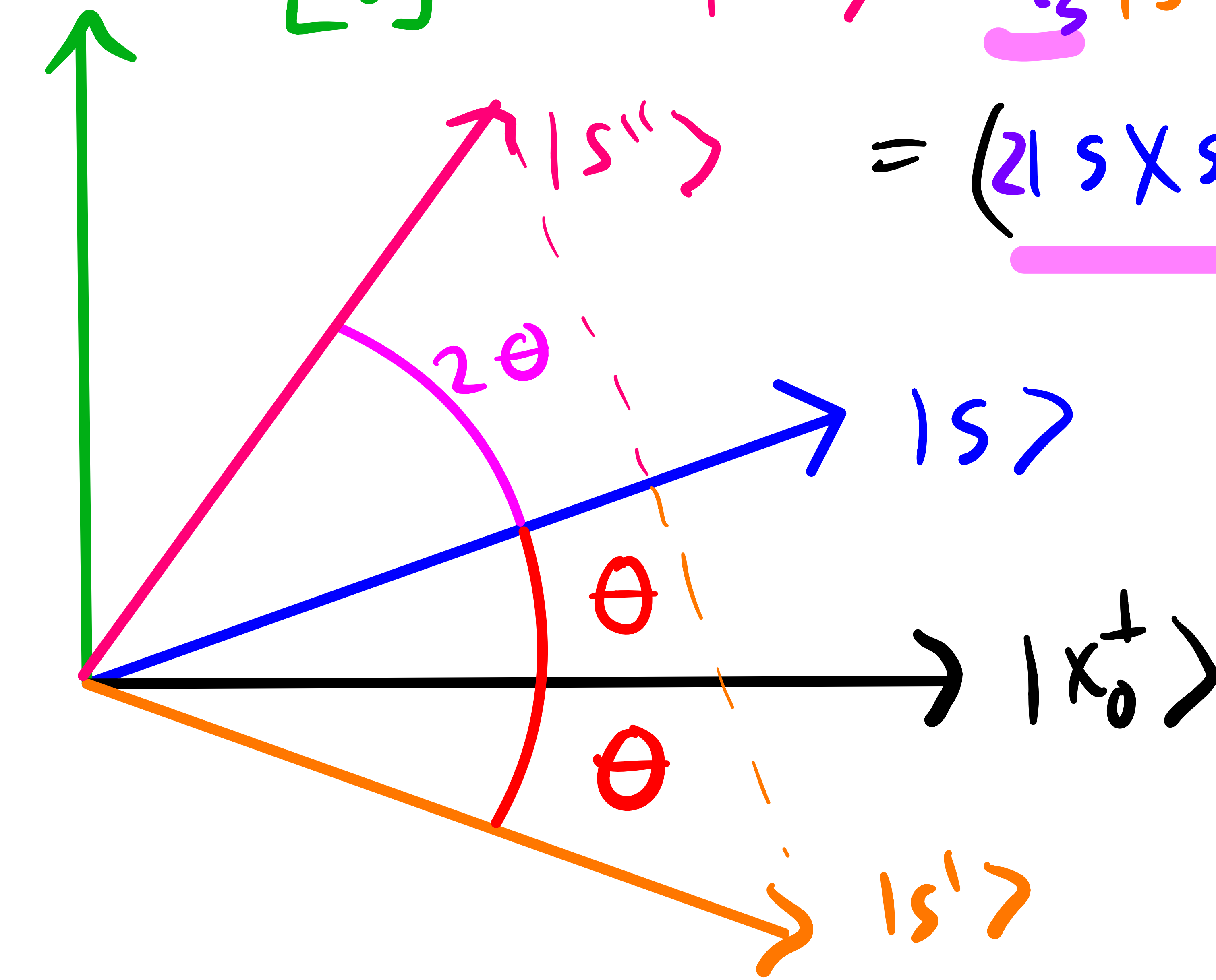
$$U_s U_w |s\rangle$$

3. Amplify

$$|x_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|s''\rangle = U_s |s'\rangle$$

$$= \left(2|s\rangle\langle s| - I\right)|s'\rangle$$



$$\left(2|s\rangle\langle s| - I\right)|s'\rangle \qquad |s\rangle = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \left(\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} - \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$$

Rat

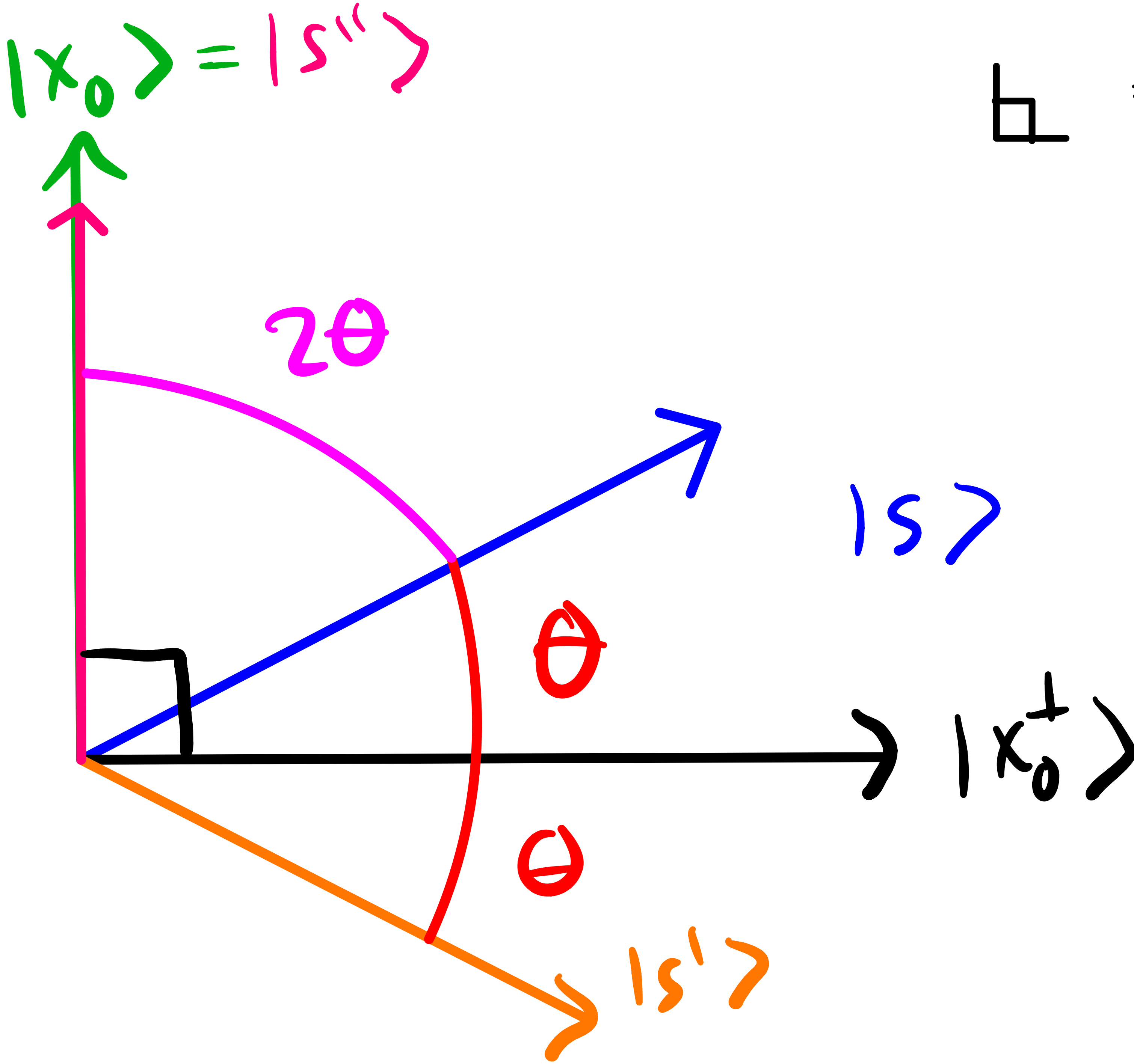
$$\left(2|s\rangle\langle s| - I\right)|s'\rangle$$

$$= \left(\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} - \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)|s'\rangle$$

$$= \frac{1}{2}\begin{array}{c}\begin{array}{cccc}1&1&-1&1\end{array}\\ \begin{bmatrix}-1&1&-1&1\\1&-1&-1&1\\1&1&+1&1\\1&1&-1&-1\end{bmatrix}\end{array} \frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}0\\0\\4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |s''\rangle$$

3. Amplify

$\llcorner = \frac{\pi}{2}$



$$\langle s | x_0^{\perp} \rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 3 \end{bmatrix}$$

$$= \frac{\sqrt{3}}{2} = |s|\,|x_0^{\perp}| \cos\theta$$

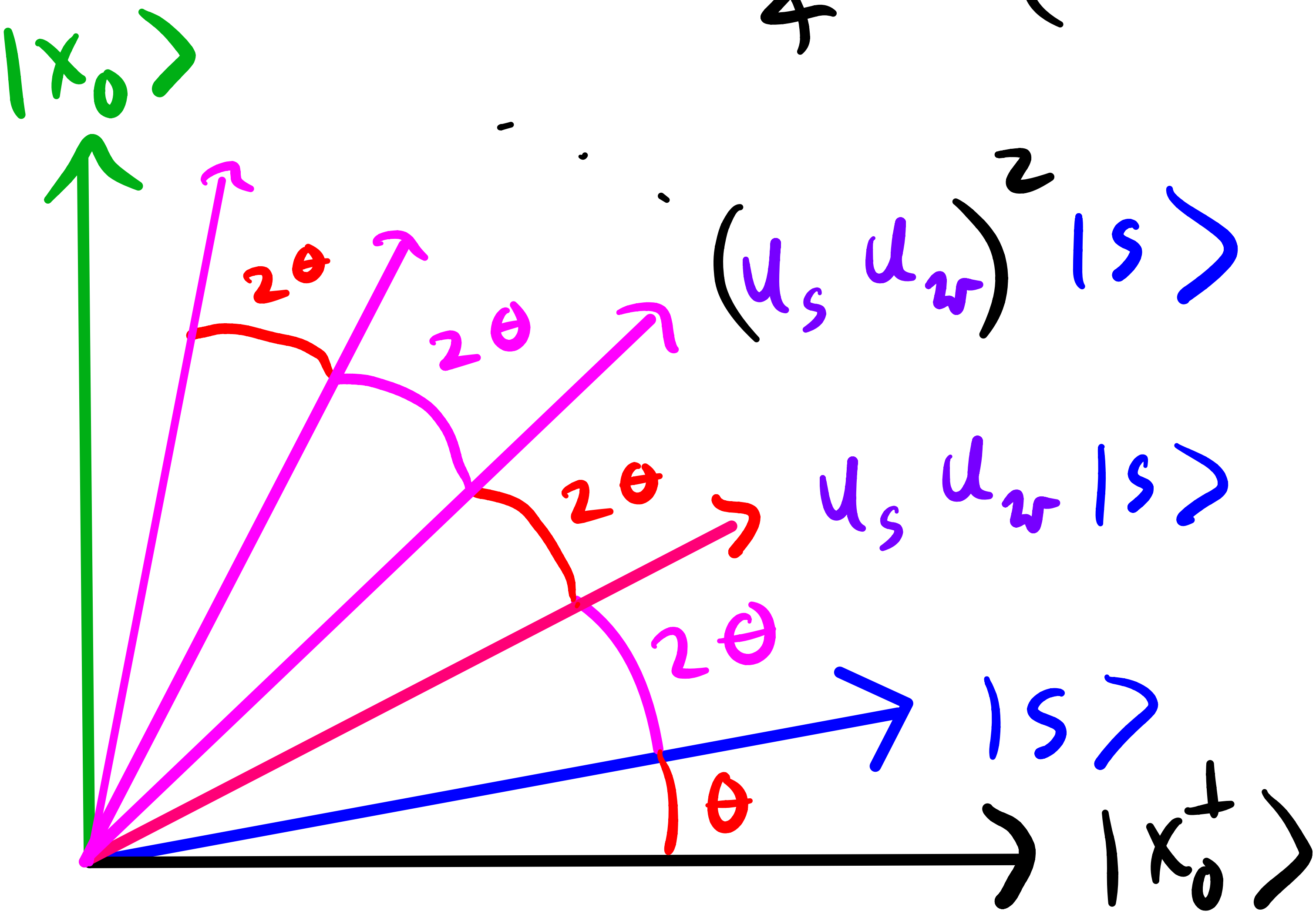
$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

In General

$$\theta = \cos^{-1}\left(\frac{\sqrt{N-1}}{\sqrt{N}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{N}}\right)$$

R iterations

$$\measuredangle = (2k+1)\theta < \frac{\pi}{2}$$



$$(2k+1)\theta < \frac{\pi}{2}$$

$$(2k+1)\sin^{-1}\left(\frac{1}{\sqrt{N}}\right) < \frac{\pi}{2}$$

$$\frac{1}{\sqrt{N}} < \sin\left(\frac{\pi}{2(2k+1)}\right)$$

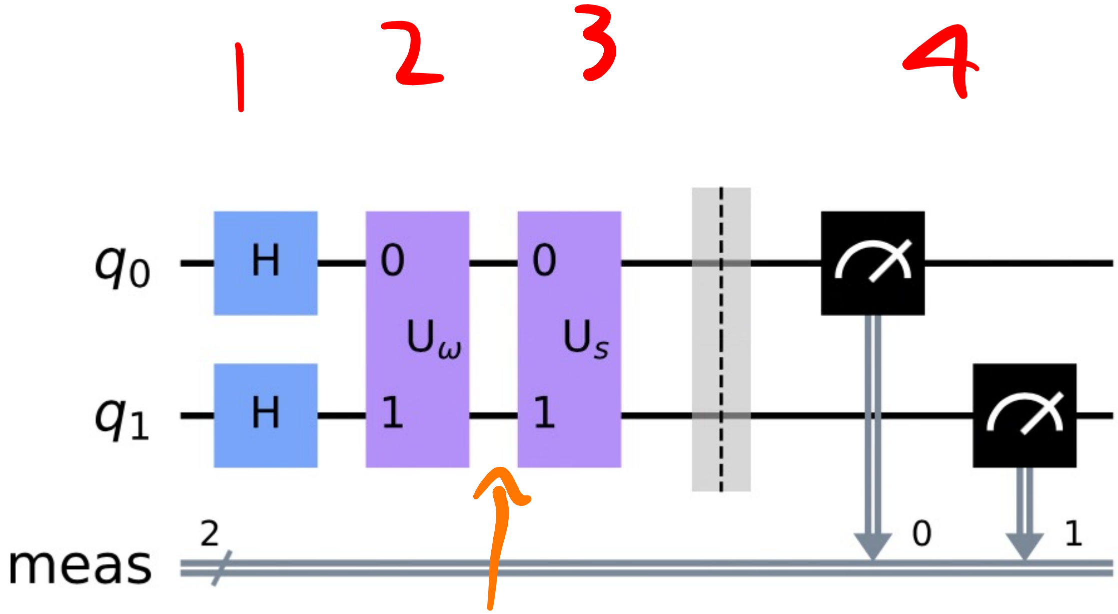
$k \gg$

$\sin\theta \approx \theta$

$$\frac{1}{\sqrt{N}} < \sin\left(\frac{\pi}{4k}\right)$$

$$\frac{1}{\sqrt{N}} < \frac{\pi}{4k}$$

$$k < \frac{\pi}{4}\sqrt{N}$$



$$|s\rangle = \frac{1}{\sqrt{4}} \sum_{x=0}^{4} |x\rangle$$

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Grover iteration

$U_s U_w |s\rangle$